Hamiltonicity of random subgraphs of the hypercube

Alberto Espuny Díaz

Technische Universität Ilmenau

joint work with Padraig Condon, António Girão, Daniela Kühn and Deryk Osthus

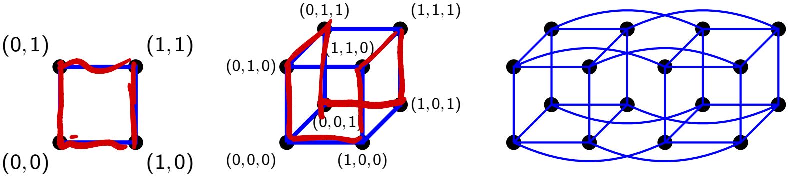
LIMDA Joint Seminar, Barcelona

11 November 2020

The hypercube

Definition

The n-dimensional hypercube Q^n:
$V(Q^n)$: set of all n-bit 01-strings.
$E(Q^n)$: $xy \in E$ iff they differ on only one position.



Theorem

For all $n \geq 2$, Q^n is Hamiltonian.

Binomial random graphs

$G_{n,p}$: n vertices, add each edge with probability p independently.

Definition

Let $\mathscr{P}$ be a monotone increasing graph property. Then, $p^* = p^*(n)$ is a *threshold* for $\mathscr{P}$ if

$$\mathbb{P}[G_{n,p} \in \mathscr{P}] \to \begin{cases} 1 & \text{if } p/p^* \to \infty, \\ 0 & \text{if } p/p^* \to 0. \end{cases}$$

Similarly, $p^* = p^*(n)$ is a *sharp threshold* for $\mathscr{P}$ if, for all $\varepsilon > 0$,

$$\mathbb{P}[G_{n,p} \in \mathscr{P}] \to \begin{cases} 1 & \text{if } p \geq (1+\varepsilon)p^*, \\ 0 & \text{if } p \leq (1-\varepsilon)p^*. \end{cases}$$

Q_p^n: delete each edge of Q^n independently with probability $1-p$.

Some thresholds

Theorem

Sharp threshold for connectivity (Erdős & Rényi, '60), containment of a perfect matching (Erdős & Rényi, '66) and Hamiltonicity (Koršunov, '76) in $G_{n,p}$: $p^* = \log n/n$.

Theorem

Sharp threshold for connectivity (Burtin, '77) and containment of a perfect matching (Bollobás, '90) in Q^n_p: $p^* = 1/2$.

Conjecture

Sharp threshold for Hamiltonicity in Q^n_p: $p^* = 1/2$.

Theorem (Condon, Espuny Díaz, Girão, Kühn, Osthus, '20$^+$)

For any $k \in \mathbb{N}$, the sharp threshold for the property of containing k edge-disjoint Hamilton cycles in Q^n_p is $p^* = 1/2$.

Graph processes

K_n, Q^n

Given an n-vertex graph G with m edges, let $\tilde{G} = (G_0, G_1, \dots, G_m)$, where G_0 is the empty graph and $G_{i+1} = G_i \cup \{e\}$, with e chosen uniformly at random among the missing edges.
Hitting time for property $\mathscr{P}$: $\tau_{\mathscr{P}}(\tilde{G}) := \min\{i \in [m] : G_i \in \mathscr{P}\}$.

Theorem
A.a.s. $\tau_{\mathrm{PM}}(\tilde{K}_n) = \tau_{\mathrm{CON}}(\tilde{K}_n) = \tau_{\delta 1}(\tilde{K}_n)$.

Theorem (Ajtai, Komlós, Szemerédi, '85; Bollobás, '84)
A.a.s. $\tau_{\mathrm{HAM}}(\tilde{K}_n) = \tau_{\delta 2}(\tilde{K}_n)$.

Theorem (Bollobás, '90)
A.a.s. $\tau_{\mathrm{PM}}(\tilde{Q}^n) = \tau_{\mathrm{CON}}(\tilde{Q}^n) = \tau_{\delta 1}(\tilde{Q}^n)$.

Conjecture
A.a.s. $\tau_{\mathrm{HAM}}(\tilde{Q}^n) = \tau_{\delta 2}(\tilde{Q}^n)$.

Graph processes

Given an n-vertex graph G with m edges, let $\tilde{G} = (G_0, G_1, \ldots, G_m)$, where G_0 is the empty graph and $G_{i+1} = G_i \cup \{e\}$, with e chosen uniformly at random among the missing edges.
Hitting time for property $\mathscr{P}$: $\tau_{\mathscr{P}}(\tilde{G}) := \min\{i \in [m] : G_i \in \mathscr{P}\}$.

Let $\mathrm{HM}k$ denote the property that G contains $\lfloor k/2 \rfloor$ Hamilton cycles and $k - 2\lfloor k/2 \rfloor$ perfect matchings, all edge-disjoint.

Theorem (Bollobás, Frieze, '85)

For any $k \in \mathbb{N}$, a.a.s. $\tau_{\mathrm{HM}k}(\tilde{K}_n) = \tau_{\delta k}(\tilde{K}_n)$.

Theorem (Condon, Espuny Díaz, Girão, Kühn, Osthus, '20$^+$)

For any $k \in \mathbb{N}$, a.a.s. $\tau_{\mathrm{HM}k}(\tilde{Q}^n) = \tau_{\delta k}(\tilde{Q}^n)$.

Randomly perturbed graphs

Union of a deterministic graph H and a random graph G.

Theorem (Bohman, Frieze, Martin, '03)

Let $\alpha > 0$. Let H be an n-vertex graph with $\delta(H) \geq \alpha n$. If $p \geq C(\alpha)/n$, then a.a.s. $H \cup G_{n,p}$ is Hamiltonian.

Theorem (Condon, Espuny Díaz, Girão, Kühn, Osthus, '20$^+$)

For all $\varepsilon, \alpha \in (0,1]$ and $k \in \mathbb{N}$, the following holds.
Let H be a spanning subgraph of Q^n such that $\delta(H) \geq \alpha n$.
Then, a.a.s. $H \cup Q^n_\varepsilon$ contains k edge-disjoint Hamilton cycles.

Lemma: $\forall \varepsilon$, $\exists \alpha$ s.t. $\delta(Q^n_{\frac{1}{2}+\varepsilon}) \geq \alpha n$

Pf: $Q^n_{\frac{1}{2}+\varepsilon} \supseteq \underbrace{Q^n_{\frac{1}{2}+\frac{\varepsilon}{2}}}_{H} \cup Q^n_{\frac{\varepsilon}{2}}$

Randomly perturbed graphs

Union of a deterministic graph and a random graph.

Theorem (Bohman, Frieze, Martin, '03)

Let $\alpha > 0$. Let H be an n-vertex graph with $\delta(H) \geq \alpha n$. If $p \geq C(\alpha)/n$, then a.a.s. $H \cup G_{n,p}$ is Hamiltonian.

Theorem (Condon, Espuny Díaz, Girão, Kühn, Osthus, '20^+)

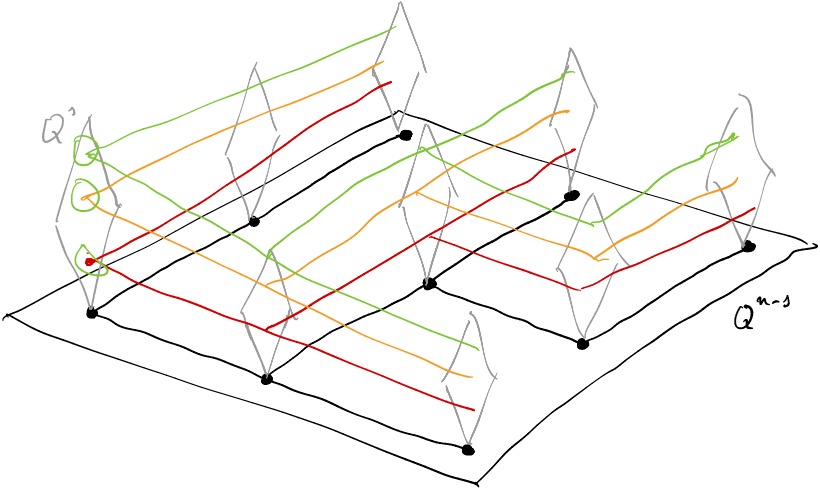
For all $\varepsilon, \alpha \in (0,1]$ and $k \in \mathbb{N}$, the following holds. Let H be a spanning subgraph of Q^n such that $\delta(H) \geq \alpha n$. Then, a.a.s. $H \cup Q^n_\varepsilon$ contains k edge-disjoint Hamilton cycles.

Theorem (Condon, Espuny Díaz, Girão, Kühn, Osthus, '20^+)

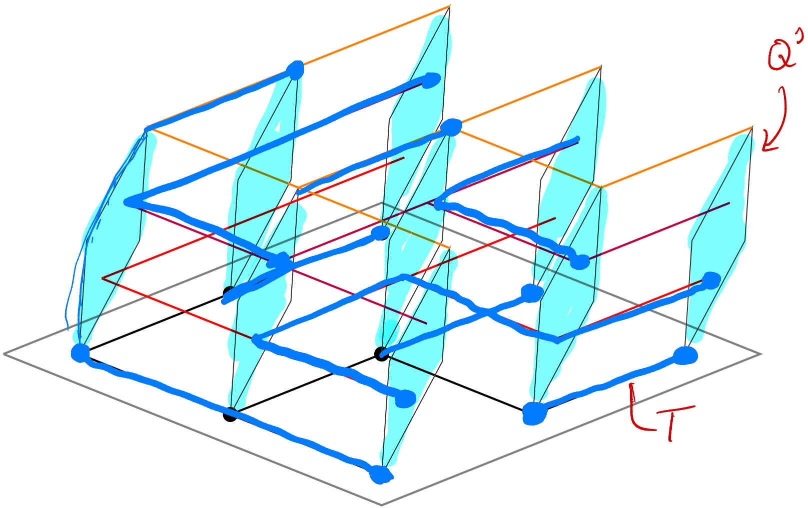
For every integer $k \geq 2$, there exists $\varepsilon > 0$ such that a.a.s., for every spanning subgraph H of Q^n with $\delta(H) \geq k$, the graph $H \cup Q^n_{1/2-\varepsilon}$ contains a collection of $\lfloor k/2 \rfloor$ Hamilton cycles and $k - 2\lfloor k/2 \rfloor$ perfect matchings, all pairwise edge-disjoint.

Proof ideas

Q^n



Proof ideas

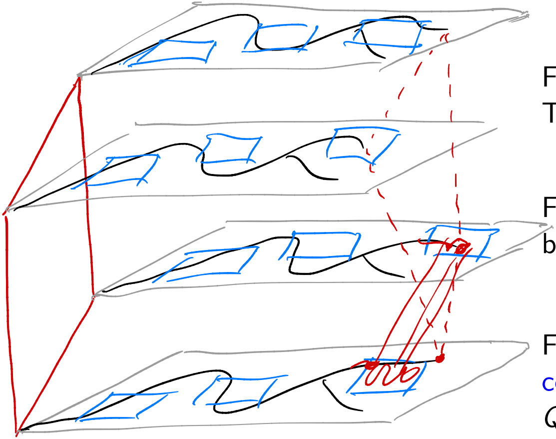


Idea: follow spanning trees in large subcubes if possible
use small subcubes to move between trees

Proof ideas

Theorem (Condon, Espuny Díaz, Girão, Kühn, Osthus, '20$^+$)

$\forall\ \delta,\varepsilon \in (0,1]$, *a.a.s.* Q^n_ε *contains a cycle of length* $\geq (1-\delta)2^n$.



Fix s coordinates.
This partitions Q^n.

Find a near-spanning bounded-degree tree.

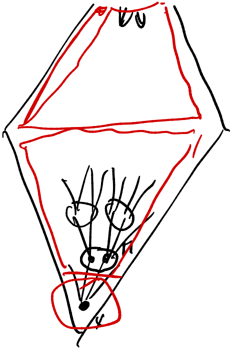
Find a near-perfect cover with copies of Q^{ℓ}. constant

Proof ideas

Theorem (Condon, Espuny Díaz, Girão, Kühn, Osthus, '20$^+$)

$\forall\ \delta, \varepsilon \in (0,1]$, *a.a.s.* Q^n_ε *contains a cycle of length* $\geq (1-\delta)2^n$.

- Near-spanning bounded-degree tree: we "grow" trees from several "corners" of the cube, following a branching process.
- Near-perfect Q^ℓ-cover: given by the Rödl nibble.

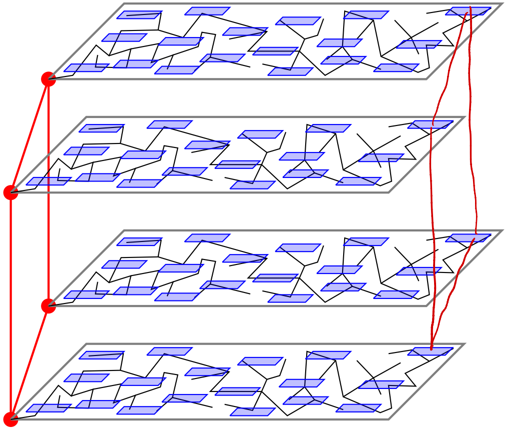


$d_{Q^n_\varepsilon}(x) \approx \varepsilon n$

Choose M

Proof ideas

Theorem (Condon, Espuny Díaz, Girão, Kühn, Osthus, '20$^+$)
$\forall\ \delta, \varepsilon \in (0,1]$, *a.a.s.* Q^n_ε *contains a cycle of length* $\geq (1-\delta)2^n$.



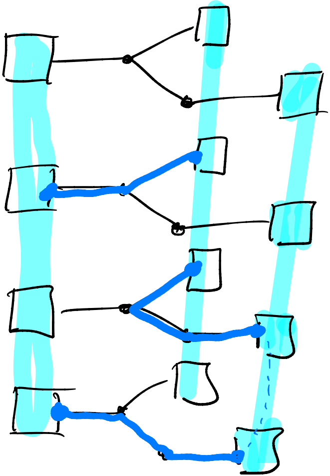
Fix s coordinates.
This partitions Q^n.

Find a near-spanning bounded-degree tree.

Find a near-perfect cover with copies of Q^ℓ.

Proof ideas

We now construct a *skeleton* for a long cycle.



→ C cycle contains all $\underline{Q^\ell}$

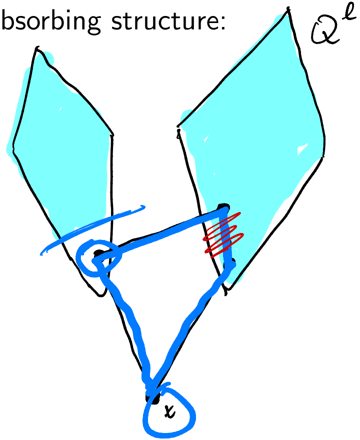
We turn this into the desired cycle with the 'connecting lemmas'.

Proof ideas

Theorem (Condon, Espuny Díaz, Girão, Kühn, Osthus, '20$^+$)

Let $\varepsilon, \alpha \in (0,1]$, $H \subseteq Q^n$, $\delta(H) \geq \alpha n$. *Then, a.a.s.* $H \cup Q^n_\varepsilon \in \mathrm{HAM}$.

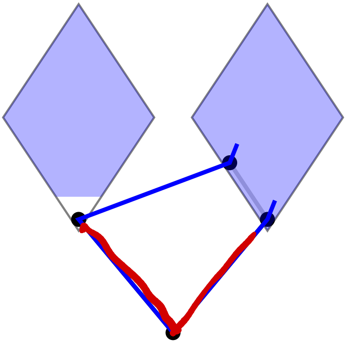
Absorbing structure:



Proof ideas

Theorem (Condon, Espuny Díaz, Girão, Kühn, Osthus, '20$^+$)
Let $\varepsilon, \alpha \in (0,1]$, $H \subseteq Q^n$, $\delta(H) \geq \alpha n$. Then, a.a.s. $H \cup Q^n_\varepsilon \in \mathrm{HAM}$.

Absorbing structure:

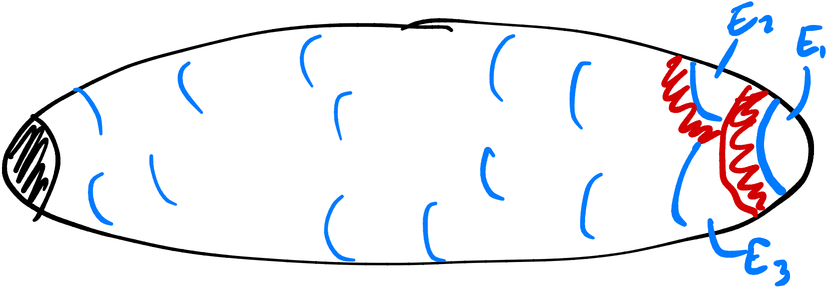


We require that

- the near-spanning tree satisfies some nice conditions (for each vertex, covers almost all neighbourhood),
- the near cover satisfies very nice conditions, some related to the graph H.

Proof ideas

The Rödl nibble



$\mathcal{H}$ nice properties

$V(\mathcal{H}) = V(Q^n)$

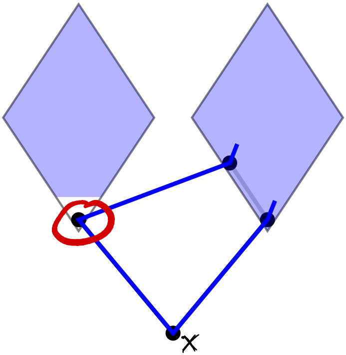
$E(\mathcal{H}) =$ copies of Q^ℓ in Q^n_ε

Proof ideas

Theorem (Condon, Espuny Díaz, Girão, Kühn, Osthus, '20$^+$)

Let $\varepsilon, \alpha \in (0,1]$, $H \subseteq Q^n$, $\delta(H) \geq \alpha n$. *Then, a.a.s.* $H \cup Q^n_\varepsilon \in \mathrm{HAM}$.

Absorbing structure:



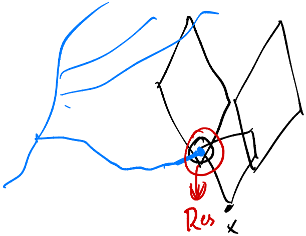
We require that

- the near-spanning tree satisfies some nice conditions (for each vertex, covers almost all neighbourhood),
- the near cover satisfies very nice conditions, some related to the graph H.

Need to deal with parity issues: vertices are absorbed in pairs.

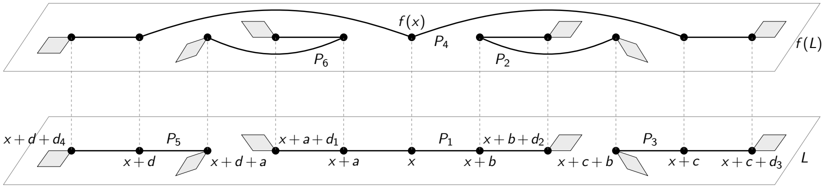
Proof ideas

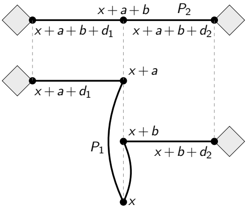
Some other problems that need ironing out

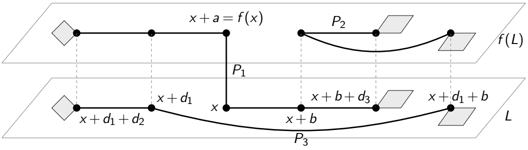


Proof ideas

For the hitting time, we need to deal with vertices of very low degree.







Open problems

Theorem (Condon, Espuny Díaz, Girão, Kühn, Osthus, '20$^+$)

$\forall\ \delta, \varepsilon \in (0,1]$, *a.a.s.* Q^n_ε *contains a cycle of length* $\geq (1-\delta)2^n$.

Conjecture

Suppose that $p = p(n)$ *satisfies that* $pn \to \infty$. *Then, a.a.s.* Q^n_p *contains a cycle of length* $(1-o(1))2^n$.

Conjecture

Suppose $\varepsilon > 0$ *and an integer* $\ell \geq 2$ *are fixed and* $p \geq 1/2 + \varepsilon$. *Then, a.a.s.* Q^n_p *contains a* C_{2^ℓ}*-factor, that is, a set of vertex-disjoint cycles of length* 2^ℓ *which together contain all vertices of* Q^n.