Minimum bisection of random 3-regular graphs

Lyuben Lichev

Joint work with Dieter Mitsche

The problem

- Graph $G(V, E)$, $|V|$ being even.
- Bisection of G: partition (V_1, V_2) of V with $|V_1| = |V_2|$.
- Size of a bisection (V_1, V_2): number of edges crossing.
- *Bisection width of G*, $\mathrm{bw}(G)$: minimum size of a bisection of G.
- Find $\mathrm{bw}(G)$.

Motivation

Wide range on applications:

- Applied mathematics.
- Computer science.
- Statistical physics.

History

A much studied problem.

- NP-complete for general graphs (Garey, Johnson, Stockmeyer, 1976).
- NP-complete for regular graphs (Bui, Chaudhuri, Leighton, Sisper, 1987).
- Lower bounds for random 3-regular graphs:
  - $n/11 \approx 0.0909n$ (Bollobás, 1984).
  - $10n/99 \approx 0.10101n$ (Kostochka and Melnikov, 1993).

History

Upper bounds are more abundant.

- $n/4 + o(n)$ for an arbitrary 3-regular graph (Kostochka and Melnikov, 1992).
- $n/6$ for an arbitrary large enough 3-regular graph (Monien and Preis, 2001).
- $0.1740n$ for a random 3-regular graph (Díaz, Do, Serna and Wormald, 2003). Weaker bound, but more general paper. Easier algorithmic approach.
- $0.16226n$ for a random 3-regular graph (Lyons, 2017).

Our contribution

Theorem (L. and Mitsche, 2020)

The bisection width of the random 3-regular graph is between $0.10329n$ and $0.13983n$ asymptotically almost surely.

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Main ideas of the proof:

Lower bound:

- precise structural characterisation of a minimum bisection,
- partially regroup these "special" bisections according to the graph they originate from,
- bound from above the number of graphs containing a "special" bisection of given size.

Our contribution

Theorem (L. and Mitsche, 2020)

The bisection width of the random 3-regular graph is between $0.10329n$ *and* $0.13983n$ *asymptotically almost surely.*

Main ideas of the proof:

Upper bound:

- local limit theory ensures convergence to the 3-regular tree T_3,
- deduce convergence of the "λ-Gaussian wave" on $G_3(n)$ to the "λ-Gaussian wave" on T_3,
- based on a well-chosen λ, find bisections of small size in $G_3(n)$.

The probability space

- The configuration model.
- A set of vertices V.
- A set of half-edges, attached to the vertices.
- Random perfect matching of the half-edges.

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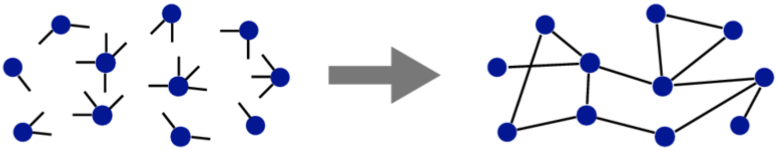

Figure: G. Zamora-López, PhD thesis.

The probability space

- The configuration model.
- A set of vertices V.
- A set of half-edges, attached to the vertices.
- Random perfect matching of the half-edges.
- This model produces a multigraph in general.
- Simple graph is sampled with probability, bounded away from 0 and 1 as $n \to +\infty$.

Winning sets

- Fix 3-regular graph G and a cut (V_1, V_2).
- Let $S \subseteq V_i$, $i \in \{1, 2\}$.
- For $\ell \geq 0$, S is ℓ-*winning* if

$$|(V_i \setminus S, V_{3-i} \cup S)| = |(V_1, V_2)| - \ell.$$

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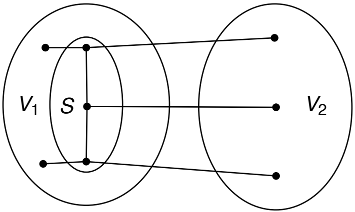

Figure: Here S is a $1-$winning set.

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- Switching winning sets S, S' with $|S| = |S'|$ and no edges between S and S' preserves the size of the parts and does not increase the cut.

Improvement

- Fix 3-regular graph G and a bisection (V_1, V_2).
- Let $S_1 \subseteq V_1$, $S_2 \subseteq V_2$ such that

$$|(V_1, V_2)| > |(S_2 \cup (V_1 \setminus S_1), S_1 \cup (V_2 \setminus S_2))|.$$

- *Improvement of* (V_1, V_2): operation of switching S_1 and S_2.

Improvements and minimum bisections

- No minimum bisection admits an improvement!
- Switching winning sets S, S' with $|S| = |S'|$, not both indifferent (i.e. not changing the size of the cut) and without edges between S and S', leads to an improvement.
- This gives information on the structure of the parts in a minimum bisection!

Typical graphs

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Almost all 3-regular graphs are typical.

Goal: bound from above the number of minimum bisections of typical 3-regular graphs of size βn, for any $\beta \geq 0.1$.

Step 1: 2-cores

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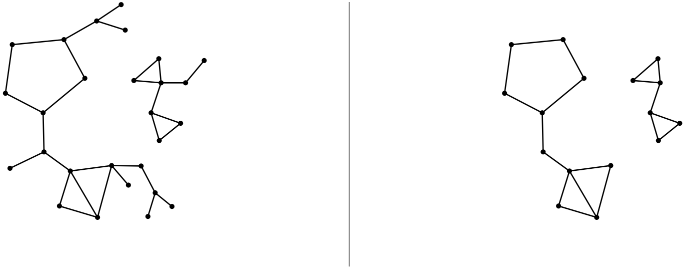

Figure: Left figure: a graph, right figure: its $2-$core

Step 1: 2-cores

Lemma

Let (V_1, V_2) *be a minimum bisection of size* βn *of a typical 3-regular graph G. Then, for both* $i = 1, 2$, $|G[V_i]| - |C_2(G[V_i])| \leq 4$.

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Sketch of proof.

We argue by contradiction. We find many paths $v_0 \dots v_k$ in $G[V_{3-i}]$, for some $k \leq 4$, such that

$$\deg_{G[V_{3-i}]}(v_0) \leq 2, \deg_{G[V_{3-i}]}(v_k) \leq 2.$$

Exchanging one of them with the set $G[V_i] \setminus C_2(G[V_i])$ would lead to an improvement. □

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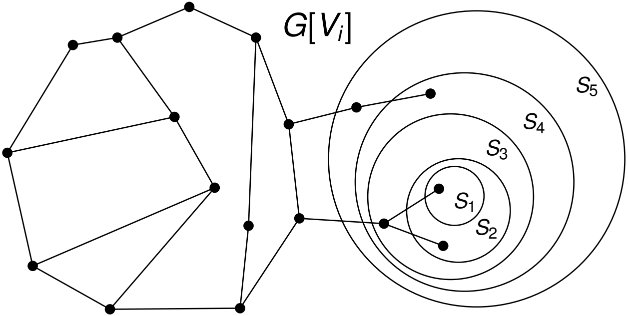

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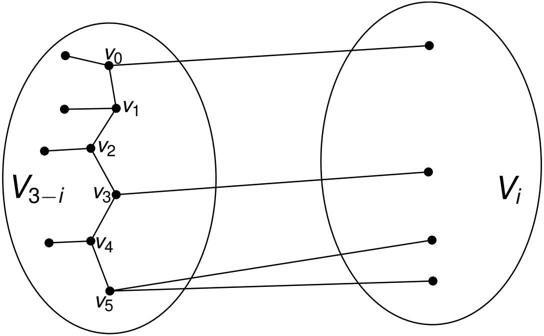
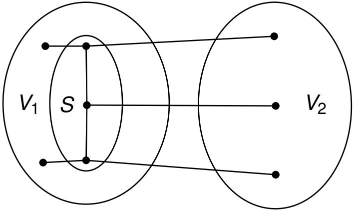


Step 2: 1-winning subgraphs

Question: Could there be many sets like S in V_1?



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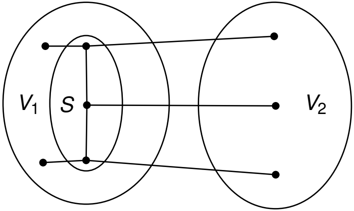

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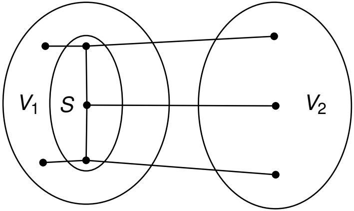

Answer: There are very few of these if there are a lot of neighbours in $G[V_2]$ of vertices of degree two.

Two cases:

- There are at most $o(n)$ pairs of neighbours of degree two in $G[V_2]$.
- There is at most one path like S in $G[V_2]$ (parity constraints).

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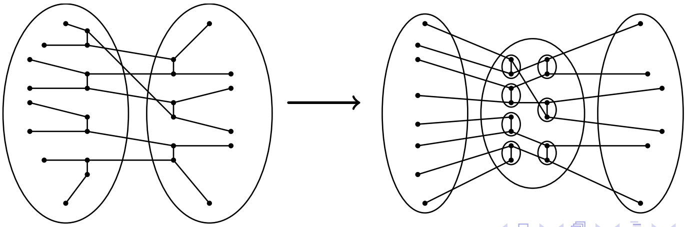
New idea: regroup "special" bisections of second type according to the graph these come from.



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Local limit theory

A *uniformly rooted graph* (G, ρ) is a finite graph with a root, chosen uniformly at random. A sequence of uniformly rooted graphs $(G_n, \rho_n)_{n \geq 1}$ *converges locally* if for every $r \geq 1$,

$$\lim_{n \to +\infty} \mathbb{P}(B_{G_n}(\rho_n, r) = (G, \rho))$$

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The limit measure of $(B_{G_n}(\rho_n, r))_{n\geq 1}$ is often abusively identified with a graph G_∞. Example: the neighbourhood of a typical vertex in $G_3(n)$ contains no cycle of constant length. Therefore, $(G_3(n))_{n\geq 1}$ converges to the 3-regular tree T_3.

Factor of iid process

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- for every $v \in V(G)$, X_v is a measurable function of $(Z_v)_{v \in V(G)}$, and
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$(X_v)_{v \in V(G)}$ is *a linear factor of iid process* on G if there exist $\alpha_0, \alpha_1, \dots$ such that $X_v = \sum_{u \in V(G)} \alpha_{d_G(v,u)} Z_u$ for all $v \in V(G)$.

Gaussian processes

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Theorem (Czóka, Gerencsér, Harangi, Virág, 2008)

For any real number λ with $|\lambda| \leq 3$ there exists a non-trivial invariant Gaussian process $(Y_v)_{v \in V(T_3)}$ on T_3 such that (with probability 1) for every vertex v

$$\sum_{u \in N(v)} Y_u = \lambda Y_v.$$

Moreover, the joint distribution of such a process is unique under the additional condition that, for every $v \in V(T_3)$, the variance of Y_v is 1. We refer to this process as the Gaussian wave function corresponding to the eigenvalue λ.

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Let $(G_n)_{n \geq 1}$ converge locally to T_3.

Theorem (Czóka, Gerencsér, Harangi, Virág, 2008)

For any real number λ with $|\lambda| \leq 2\sqrt{2}$ there exists a sequence of linear factors of iid processes $(X_v^{(n)})_{v \in V(G_n)}$ that converges in distribution to the Gaussian wave function $(Y_v)_{v \in V(T_3)}$ corresponding to the eigenvalue λ.

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Idea (Lyons, 2017): Form a cut of the vertices of $G_3(n)$ according to the signs of $(X_v)_{v\in V(G_3(n))}$.

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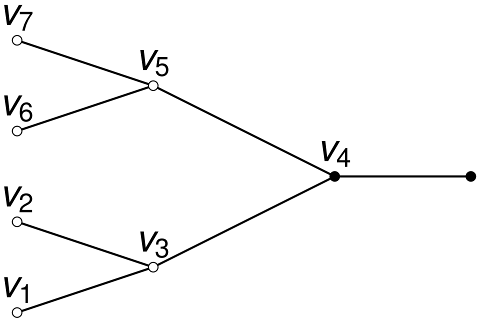

Exact results

- A study of the above structure leads to an improved upper bound of $0.1398n$ on the bisection width.
- Studying longer chains of repeating black and white vertices is out of reach for us due to numerical difficulties. Monte-Carlo simulations allow us to make a non-rigorous prediction of an upper bound of $0.1303n$.

Thank you for your attention!

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