Optimization of eigenvalue bounds for the independence and chromatic number of graph powers

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(Joint work with A. Abiad, G. Coutinho, B. D. Nogueira, and S. Zeijlemaker)

Talk based on the following papers:

1. A. Abiad, S. M Cioabă and M. Tait, Spectral bounds for the k-independence number of a graph, *Linear Algebra Appl.* **510** (2016) 160–170.
2. A. Abiad, G. Coutinho and M. A. Fiol, On the k-independence number of graphs, *Discrete Math.* **342** (2019), no. 10, 2875–2885.
3. M. A. Fiol, A new class of polynomials from the spectrum of a graph, and its application to bound the k-independence number, *Linear Algebra Appl.* **605** (2020) 1–20.
4. A. Abiad, G. Coutinho, M. A. Fiol, B. Nogueira and S. Zeijlemaker, Optimization of eigenvalue bounds for the independence and chromatic number of graph powers, `arXiv:2010.12649`.

Abstract

The k^{th} power G^k of a graph $G = (V, E)$ is the graph whose vertex set is V, and in which two distinct vertices are adjacent if and only if their distance in G is at most k. In this talk we present various spectral bounds for the k-independence number and k-chromatic number, together with a method to optimize them.
In particular, such bounds are shown to be tight for some of the so-called k-partially walk-regular, which can be seen as a generalization of distance-regular graphs. In this case, the bounds are obtained via a new family of polynomials obtained from the spectrum of a graph, called minor polynomials. Moreover these results coincide with the Delsarte's linear programming bound and, in fact, the given bounds also apply also for theLovász theta number θ, and the Shannon capacity of a graph Θ.
In some cases, our approach has the advantage of yielding closed formulas and, so, allowing asymptotic analysis. In some cases, we use LP and MILP to optimize the results.

1. Preliminaries and some previous work

Graphs and spectra

Let $G = (V, E)$ be a graph with $n = |V|$ vertices, $m = |E|$ edges, and adjacency matrix $\boldsymbol{A}$ with spectrum

$$\operatorname{sp} G = \{\theta_0, \theta_1^{m_1}, \cdots, \theta_d^{m_d}\}.$$

where $\theta_0 > \theta_1 > \cdots > \theta_d$.
When the eigenvalues are presented with possible repetitions, we shall indicate them by

$$\operatorname{ev} G : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Let us consider the scalar product in $\mathbb{R}_d[x]$:

$$\langle f, g \rangle_G = \frac{1}{n} \operatorname{tr}(f(\boldsymbol{A}) g(\boldsymbol{A})) = \frac{1}{n} \sum_{i=0}^{d} m_i f(\theta_i) g(\theta_i).$$

The predistance polynomials $p_0 (= 1), p_1, \ldots, p_d$ are a sequence of orthogonal polynomials with respect to the above product, with $\operatorname{dgr} p_i = i$, normalized in such a way that $\|p_i\|_G^2 = p_i(\theta_0)$

Closed walks and adjacency matrix

Adjacency matrix $\boldsymbol{A} = (a_{ij})$

Power adjacency matrix $\boldsymbol{A}^k = (a^k_{ij})$

$$a^k_{ij} = \#\ \text{walks of length k from i to j}$$

Graph powers

Given a graph G, the k-th power G^k is formed from G by adding all edges between vertices at distance $\leq k$.

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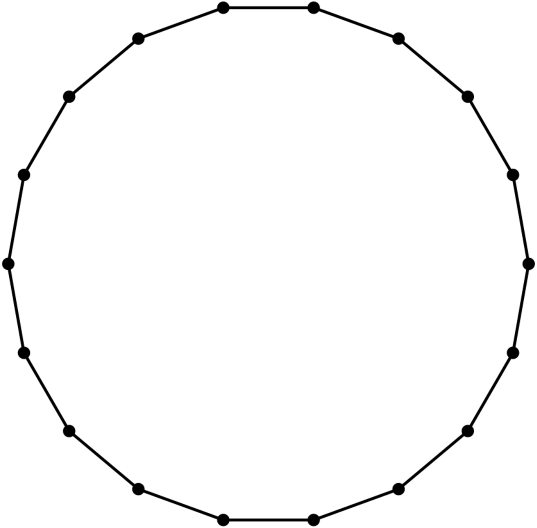
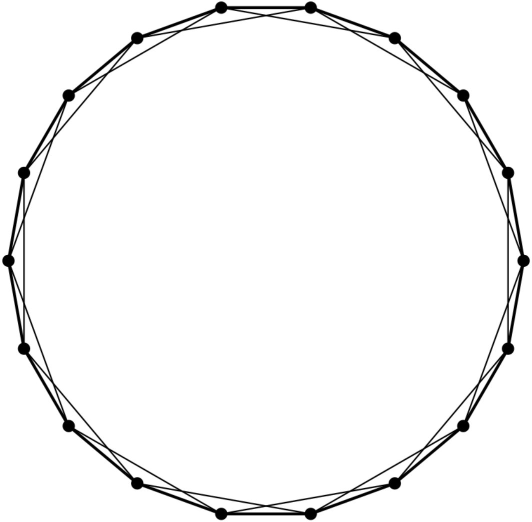


G

Graph powers

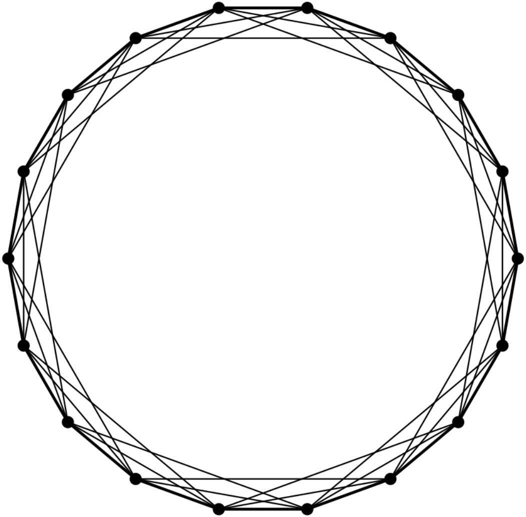
Given a graph G, the k-th power G^k is formed from G by adding all edges between vertices at distance $\le k$.



G^2

Graph powers

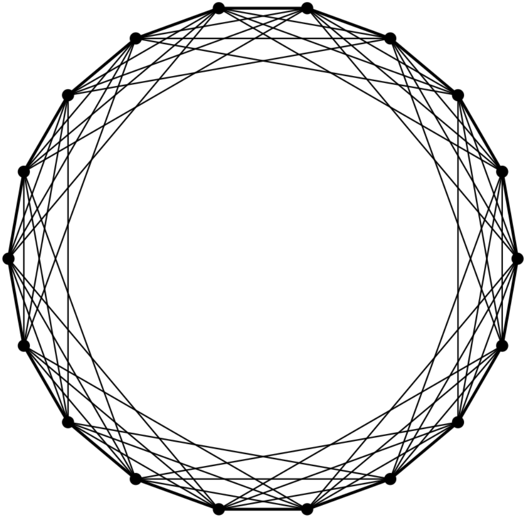
Given a graph G, the k-th power G^k is formed from G by adding all edges between vertices at distance $\leq k$.



G^3

Graph powers

Given a graph G, the k-th power G^k is formed from G by adding all edges between vertices at distance $\leq k$.



G^4

Independence number

Independence number $\alpha(G)$: size of the largest independent set of vertices in G

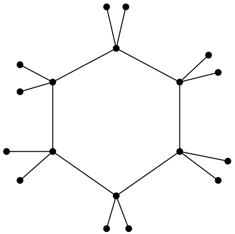
k-independence number

k-**independence number** $\alpha_k(G)$: maximum size of a set of vertices at pairwise distance greater than k

$$\alpha_k(G) = \alpha(G^k)$$

k-independence number

α_2 of the Corona graph?



k-independence number

$\alpha_0(G)$: number of vertices of G

$\alpha_1(G)$: independence number of G

Some previous work on α_k

(Kong and Zhao 1993) For every $k \geq 2$, determining $\alpha_k(G)$ is **NP-complete** for general graphs.

(Kong and Zhao 2000) The problem remains NP-complete for regular bipartite graphs when $k \in \{2, 3, 4\}$.

(Duckworth and Zito 2003) Heuristic-based algorithm for the 2-independence number.

Some previous work on α_k

(Firby and Haviland 1997) Upper bound for $\alpha_k(G)$ in an n-vertex connected graph.

(F. 1997) Eigenvalue upper bound for α_k using alternating polynomials.

(Beis, Duckworth and Zito 2005) For each fixed integer $k \geq 2$ and $r \geq 3$, upper bounds for $\alpha_k(G)$ in random r-regular graphs.

Some previous work on α_k

(O, Shi and Taoqiu 2019) Sharp upper bounds for the k-independence number in an n-vertex r-regular graph for each positive integer $k \geq 2$ and $r \geq 3$.

(Jou, Lin and Lin 2020) Sharp upper bound for the 2-independence number of a tree.

(Enami and Negami 2020) The k-independence number is related to the beans function of a connected graph.

Relation to coding theory

- **Coding theory**

Codes are k-independent sets in Hamming graphs

Spectral bounds for α_k used to show the non existence of certain perfect codes in Odd graphs (F. 2020)

Relation with other combinatorial parameters

- **average distance** (Firby and Haviland 1997): The k-independence number can be used to formulate sharp lower bounds for the average distance.
- d**-diameter** (Chung, Delorme and Solé 1999): A h-code in a graph G with distance d is a set of $h \geq 2$ vertices with $\min_{i \neq j}(d(x_i, x_j)) = d$. The d-diameter of G, say D_h, is the largest possible distance a h-code in G can have. D_2 is the standard diameter.

k-distance chromatic number

k-**distance chromatic number** $\chi_k(G)$:

$$\chi_k(G) = \chi(G^k)$$

Since $\chi(G) \geq n/\alpha(G)$, upper bounds on the k-independence number give lower bounds on the k-distance chromatic number, and vice versa.

Previous work on χ_k: a class of colouring problem

Question (Alon and Mohar 2002)
What is the largest possible value of the chromatic number $\chi(G^k)$ of G^k, among all graphs G with maximum degree at most d and girth at least g?

Case $k = 1$ (long-standing problem of Vizing): settled asymptotically by (Johansson 1996) using the probabilistic method.

Case $k = 2$: settled asymptotically by (Alon and Mohar 2002).

General k: bounds by (Alon and Mohar 2002), (Kang and Pirot 2016), (Kang and Pirot 2018), ...

Our goal

Extension of two classical bounds for the independence number of a graph.

First classical spectral bound for α

(Cvetković's inertia bound 1972)
If G is a graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, then

$$\begin{aligned}\alpha(G) &\leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\} \\ &= \min\{N^+, N^-\}.\end{aligned}$$

Second classical spectral bound for α

(Hoffman's ratio bound 1970)
If G is regular with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, then

$$\alpha(G) \leq n \frac{-\lambda_n}{\lambda_1 - \lambda_n}.$$

and if an independent set C meets this bound then every vertex not in C is adjacent to precisely $-\lambda_n$ vertices of C.

Delsarte proved the ratio bound for SRGs, later Hoffman extended it to regular graphs.

Motivation

α_k is the independence number α of G^k

BUT

Even the simplest spectral or combinatorial parameters of G^k cannot be deduced easily from the parameters of G...
thus one cannot just apply the inertia or ratio bound on G^k.

$\Downarrow$

Our bounds depend only on the parameters (eigenvalues) of the original graph G and not of G^k

Lovász theta function

(Lovász 1979)
Lovász theta function ϑ of G provides an upper bound for the independence number α of G:

$$\alpha(G) \leq \vartheta(G).$$

Therefore the Lovász theta function ϑ of G^k, $\vartheta(G^k) = \vartheta_k(G)$, provides an upper bound for the independence number of G^k, $\alpha(G^k) = \alpha_k(G)$:

$$\alpha_k(G) \leq \vartheta_k(G).$$

Hoffman-type bound vs Lovász theta bound

Lovász theta number ϑ of G^k, $\vartheta_k(G)$, upper bounds $\alpha_k(G)$ ...

$$\alpha_k(G) \leq \vartheta_k(G)$$

but also

(Lovász 1979)

$$\vartheta \leq \text{Hoffman's bound}$$

Our Hoffman-type bound cannot beat the Lovász theta number of G^k.
However, computing our eigenvalue bounds (MILPs) are, for small graphs, faster than solving an SDP, and in many cases our bounds perform fairly good.

2. Some key concepts and results

A graph G is called *k-partially walk-regular*, for some integer $k \geq 0$, if the number of closed walks of a given length $l \leq k$, rooted at a vertex v, only depends on l.

Thus, every (simple) graph is k-partially walk-regular for $k = 0, 1$, and every regular graph is 2-partially walk-regular.

Moreover G is k-partially walk-regular for any k if and only if G is walk-regular, a concept introduced by Godsil and McKay (1980).

For example, it is well-known that every distance-regular graph is walk-regular (but the converse does not hold).

Eigenvalue interlacing

Given square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ with respective eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_m$, with $m < n$, we say that the second sequence *interlaces* the first if, for all $i = 1, \ldots, m$, it follows that

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}.$$

Theorem (Haemers, 1995; F., 1999)

Let $\boldsymbol{S}$ be a real $n \times m$ matrix such that $\boldsymbol{S}^T\boldsymbol{S} = \boldsymbol{I}$, and let $\boldsymbol{A}$ be a $n \times n$ matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Define $\boldsymbol{B} = \boldsymbol{S}^T\boldsymbol{A}\boldsymbol{S}$, and call its eigenvalues $\mu_1 \geq \cdots \geq \mu_m$. Then,

(i) *The eigenvalues of $\boldsymbol{B}$ interlace those of $\boldsymbol{A}$.*

$\vdots$

(ii)

Two interesting cases for the matrix $\boldsymbol{S}$

Let $\boldsymbol{A}$ be the adjacency matrix of a graph $G = (V, E)$.

- First, if $\boldsymbol{B}$ is a principal submatrix of $\boldsymbol{A}$, then $\boldsymbol{B}$ corresponds to the adjacency matrix of an induced subgraph G' of G.
- Second, when, for a given partition of the vertices of Γ, say $V = U_1 \cup \cdots \cup U_m$, $\boldsymbol{B}$ is the so-called *quotient matrix* of $\boldsymbol{A}$, with elements b_{ij}, $i, j = 1, \ldots, m$, being the average row sums of the corresponding block $\boldsymbol{A}_{ij}$ of $\boldsymbol{A}$.

The minor polynomials

Let $G = (V, E)$ be a graph with $\operatorname{sp} G = \{\theta_0 > \theta_1^{m_1} > \cdots > \theta_d^{m_d}\}$. The k-minor polynomial $p_k \in \boldsymbol{R}_k[x]$ is the polynomial defined by $p_k(\theta_0) = 1$ and $p_k(\theta_i) = x_i$, $1 \leq i \leq d$, where the vector $(x_1, x_2, \ldots, x_d)$ is a solution of the following linear programming problem:

$$\boxed{\begin{array}{rl} \texttt{minimize} & \sum_{i=0}^{d} m_i p(\theta_i) \\ \texttt{with constraints} & f[\theta_0, \ldots, \theta_m] = 0,\ m = k+1, \ldots, d \\ & x_i \geq 0,\ i = 1, \ldots, d, \end{array}}$$

where $f[\theta_0, \ldots, \theta_m]$ denote the m-th divided differences of Newton interpolation, recursively defined by
$f[\theta_i, \ldots, \theta_j] = \frac{f[\theta_{i+1}, \ldots, \theta_j] - f[\theta_i, \ldots, \theta_{j-1}]}{\theta_j - \theta_i}$, where $j > i$, starting with
$f[\theta_i] = p_k(\theta_i) = x_i$, $0 \leq i \leq d$.

Thus, we can easily compute the minor polynomial by using the simplex method. Moreover, as the problem is in the so-called standard form, with d variables $(x_1, \ldots, x_d)$ and $d - (k+1) + 1 = d - k$ equations, the 'basic vectors' have at least $d - (d - k) = k$ zeros.

In fact, p_k has degree k, with exactly k zeros at the mesh $\theta_1, \ldots, \theta_d$. This fact, together with $p_k(\theta_0) = 1$ and $p_k(\theta_i) \geq 0$ for $i = 1, \ldots, d$ drastically reduces the number of possible candidates for p_k.

Some particular values of k

- The cases $k = 0$ and $k = d$ are easy. Clearly, $p_0 = 1$, and p_d has zeros at all the points θ_i for $i \neq 0$. In fact, $p_d = \frac{1}{n}H$, where H is the Hoffman polynomial (1963).
- For $k = 1$, the only zero of p_1 must be at θ_d. Hence $p_1(x) = \frac{x-\theta_d}{\theta_0-\theta_d}$.
- For $k = 2$, the two zeros of p_2 must be at consecutive eigenvalues θ_i and θ_{i-1}. More precisely, θ_i must be the largest eigenvalues not greater than -1. Then, $p_2(x) = \frac{(x-\theta_i)(x-\theta_i)}{(\theta_0-\theta_i)(\theta_0-\theta_i)}$.
- When $k = 3$, the only possible zeros of p_3 are θ_d and the consecutive pair θ_i, θ_{i-1} for some $i \in [2, d-1]$. In this case, such a pair seems be around the 'center' of the mesh (see the examples below).
- When $k = d - 1$, the polynomial p_{d-1} takes only one non-zero value at the mesh, say at θ, which seems to be located at one of the 'extremes' of the mesh (e.g., when G is an r-antipodal distance-regular graph, the choice $\theta = \theta_1$ yields the tight bound (that is, r) for α_{d-1}.

The case of the Hamming graph $H(2,7)$

This is an antipodal distance-regular graph, with $n = 128$ vertices, diameter $D = 7$, and spectrum

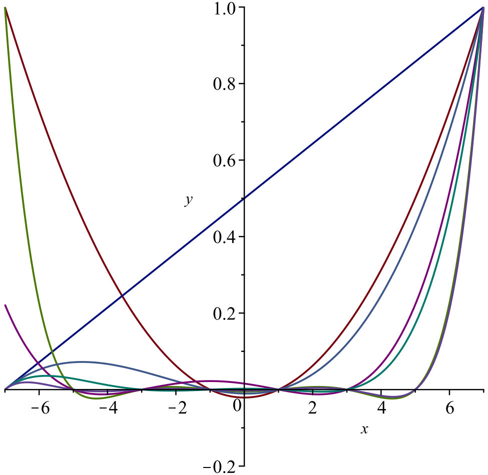
$$\operatorname{sp} H(2,7) = \{7^1, 5^7, 3^{21}, 1^{35}, -1^{35}, -3^{21}, -5^7, -7^1\}.$$

Then the solutions of the linear programming problem are:

k	x_7	x_6	x_5	x_4	x_3	x_2	x_1	x_0
1	0	1/7	2/7	3/7	4/7	5/7	6/7	1
2	1	1/2	1/6	0	0	1/6	1/2	1
3	0	1/14	1/21	0	0	5/42	3/7	1
4	2/9	0	0	1/45	0	0	2/9	1
5	0	1/35	0	0	0	0	6/35	1
6	1	0	0	0	0	0	0	1
7	0	0	0	0	0	0	0	1

Table: Values $x_i = p_k(\theta_i)$ of the k-minor polynomials of the Hamming graph $H(2,7)$.

The minor polynomials of the Hamming graph $H(2,7)$



The JOHNSON GRAPH $J(14,7)$

This is an antipodal (but not bipartite) distance-regular graph, with $n = 3432$ vertices, diameter $D = 7$, and spectrum

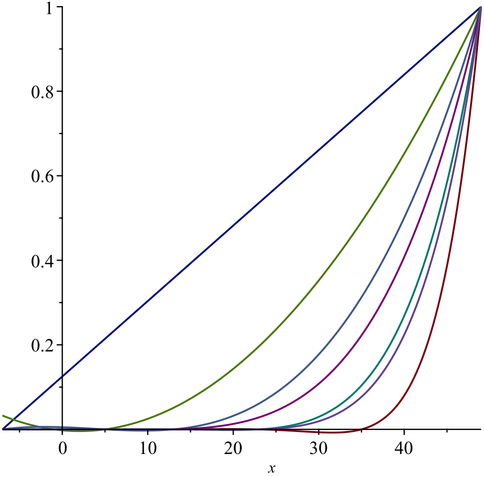
$$\operatorname{sp} J(14,7) = \{49^1, 35^{13}, 23^{77}, 13^{273}, 5^{637}, -1^{1001}, -5^{1001}, -7^{429}\}.$$

Then the solutions of the linear programming problem are:

k	x_7	x_6	x_5	x_4	x_3	x_2	x_1	x_0
1	0	1/28	3/28	3/14	5/15	15/28	3/4	1
2	9/275	1/55	0	0	14/275	54/275	27/55	1
3	0	5/1232	1/176	0	0	75/1232	5/16	1
4	1/1485	0	0	0	0	14/495	2/9	1
5	0	1/2860	0	0	0	0	27/260	1
6	0	0	0	0	0	0	1/13	1
7	0	0	0	0	0	0	0	1

Table: Values $x_i = p_k(\theta_i)$ of the k-minor polynomials of the Johnson graph $J(14,7)$.

The minor polynomials of the Johnson graph graph $J(14, 7)$



3. Our first main results

Recall our inspiring results for α

Theorem (Cvetković, 1971)

Let G be a graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then,

$$\alpha \leq \min\{|\{i : \lambda_i \geq 0\}|, |\{i : \lambda_i \leq 0\}|\}.$$

Theorem (Hoffman, 1995)

If G is a regular graph on n vertices with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, then

$$\alpha \leq \frac{n}{1 - \frac{\lambda_1}{\lambda_n}}. \tag{1}$$

With the alternating polynomials

Let G be a graph with distinct eigenvalues $\theta_0 > \cdots > \theta_d$. The k-**alternating polynomial** $P_k(x)$ of G is chosen among all polynomials $p(x) \in \mathbb{R}_k(x)$ satisfying $|p(\theta_i)| \leq 1$ for all $i = 1, \ldots, d$ and such that $P_k(\theta_0)$ is maximized. P_k was shown to be unique (F., Garriga, Yebra, 1996).

Theorem (F., 1997)

Let G be a d-regular graph on n vertices, with distinct eigenvalues $\theta_0 > \cdots > \theta_d$ and let $P_k(x)$ be its k-alternating polynomial. Then,

$$\alpha_k \leq \frac{2n}{P_k(\theta_0) + 1}. \tag{2}$$

An Inertial-type bound

$w_k(G) := \min_i (A^k)_{ii}$ (minimum number of closed walks of length k taken over all V)

$W_k(G) := \max_i (A^k)_{ii}$ (maximum number of closed walks of length k taken over all V)

(Abiad, Cioabă and Tait 2016)
Let G be a graph on n vertices. Then,

$$\alpha_k(G) \leq |\{i : \lambda_i^k \geq w_k(G)\}| \quad \text{and} \quad \alpha_k(G) \leq |\{i : \lambda_i^k \leq W_k(G)\}|.$$

Corollary Inertial-type bound, $k = 1$

(Cvetković 1972)
If G is a graph, then

$$\alpha(G) \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}.$$

A Hoffman-type bound

(n,d,λ)-**graph**: d-regular graph on n vertices with $d = \lambda_1 \geq \cdots \geq \lambda_n \geq -d$ (the d-regularity of the graph guarantees that its largest-magnitude eigenvalue is d) and $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$

$\tilde{W}_k := \max_i \sum_{j=1}^{k} (A^j)_{ii}$ (maximum number of closed walks of length **at most** k over all V)

(Abiad, Cioabă and Tait 2016)
Let G be an (n,d,λ)-graph and k a natural number. Then

$$\alpha_k(G) \leq n \frac{\tilde{W}_k + \sum_{j=1}^{k} \lambda^j}{\sum_{j=1}^{k} d^j + \sum_{j=1}^{k} \lambda^j}.$$

Corollary Hoffman-type bound, $k = 1$

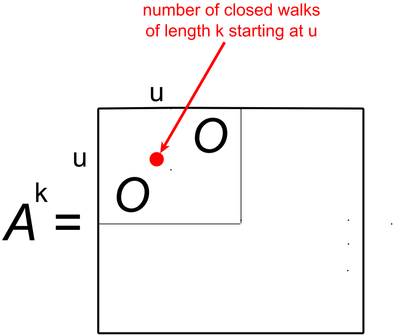
(Hoffman 1970)
If G is regular then

$$\alpha(G) \leq n \frac{-\lambda_n}{\lambda_1 - \lambda_n}.$$

Proof idea

G has a k-independent set U of size α_k.

The matrix A^k has a principal submatrix of size α_k, which is indexed by the vertices in U, and whose off-diagonal entries are 0 and whose diagonal entries equal the number of closed walks of length k starting at vertices of U.



Cauchy interlacing leads to the result.

□

4. Our latest main results

4.1 The spectrum of G^k and G are related.

4.2 The case when G is partially walk-regular.

4.3 The general case (the spectrum of G^k and G are NOT related).

4.1 The spectrum of G^k and G are related

... when the adjacency matrix of G^k belongs to the algebra generated by the adjacency matrix of $\boldsymbol{A}$, i.e., $\exists$ a polynomial p such that

$$p(\boldsymbol{A}(G)) = \boldsymbol{A}(G^k).$$

For instance, when G is k-partially distance polynomial (Dalfó, van Dam, F., Garriga and Gorissen 2011)

4.1 The spectrum of G^k and G are related

... when the adjacency matrix of G^k belongs to the algebra generated by the adjacency matrix of $\boldsymbol{A}$, i.e., $\exists$ a polynomial p such that

$$p(\boldsymbol{A}(G)) = \boldsymbol{A}(G^k).$$

Question (Alon and Mohar 2002)
What is the largest possible value of the chromatic number $\chi(G^k)$ of G^k, among all graphs G with maximum degree at most d and girth at least g?

4.1 The spectrum of G^k and G are related

(F. 2012)

Let $G = (V, E)$ be a regular graph with n vertices, spectrum $\{\theta_0^{m_0}, \theta_1^{m_1}, \dots, \theta_d^{m_d}\}$, and predistance polynomials $p_0, \dots, p_d$. For a given integer $k \leq d$ and a vertex $u \in V$, let $s_k(u)$ be the number of vertices at distance at most k from u, and consider the sum polynomial $q_k = p_0 + \cdots + p_k$. Then, $q_k(\theta_0)$ is bounded above by the harmonic mean H_k of the numbers $s_k(u)$, that is

$$q_k(\theta_0) \leq H_k = \frac{n}{\sum_{u \in V} \frac{1}{s_k(u)}},$$

and equality occurs if and only if $q_k(A) = I + A(G^k)$.

4.1The spectrum of G^k and G are related

Using that $q_k(\theta_0) \geq q_k(\theta_i)$ for $i = 1, \ldots, d$, (F. 2012) and the inertia and ratio bounds:

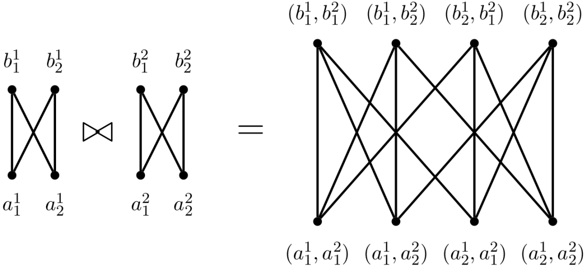
(Abiad, Coutinho, F., Nogueira and Zeijlemaker 2020)
Let G be a regular graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, satisfying $q_k(\lambda_1) = H_k$. Let $q'_k = q_k - 1$, so that $A(G^k) = q'_k(A)$. Then,

$$\chi_k \geq \frac{n}{\min\{|\{i : q'_k(\lambda_i) \geq 0\}|, |\{i : q'_k(\lambda_i) \leq 0\}|\}},$$

$$\chi_k \geq \frac{n}{1 - \frac{q'_k(\lambda_1)}{\min\{q'_k(\lambda_i)\}}}.$$

4.1 The spectrum of G^k and G are related

(Kang and Pirot 2016) provided a lower bound for χ_k from a direct construction whose building blocks are incidence structures. Those graphs which attain equality in our Hoffman-type bound.



4.1 The spectrum of G^k and G are related

Another case where $\boldsymbol{A}(G^k) = q_k(\boldsymbol{A}) - \boldsymbol{I}$ (spectra of G^k and G are related) is when G is δ-regular graph with girth g and $k = \lfloor \frac{g-1}{2} \rfloor$.

In this situation, G is k-**partially distance-regular** with $c_i = 1$ $(1 \le i \le k)$, $a_i = 0$ $(0 \le i \le k-1)$, $b_0 = \delta, b_i = \delta - 1$ $(1 \le i \le k-1)$ (Dalfó, van Dam, F., Garriga and Gorissen 2011), (Abiad, van Dam and F. 2016) and $q_0 = 1$, $q_1 = 1 + x$, $q_{i+1} = xq_i - (\delta - 1)q_{i-1}$ for $i = 1, \ldots, k-1$.

Name	Girth	k	α_k
Moebius-Kantor Graph	6	2	4
Nauru Graph	6	2	6
Blanusa First Snark Graph	5	2	4
Blanusa Second Snark Graph	5	2	4
Brinkmann graph	5	2	3
Heawood graph	6	2	2
Sylvester Graph	5	2	6
Coxeter Graph	7	3	4
Dyck graph	6	2	8
F26A Graph	6	2	6
Flower Snark	5	2	5

4.2 The case when G is partially walk-regular

Theorem (F. 2020)

Let G be a k-partially walk-regular graph with n vertices, adjacency matrix $\boldsymbol{A}$, and spectrum $\operatorname{sp} G = \{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$. Let $p_k \in \mathbb{R}_k[x]$ be the k-minor polynomial. Then,

$$\alpha_k \leq \operatorname{tr} p_k(\boldsymbol{A}) = \sum_{i=0}^{d} m_i p_k(\theta_i). \tag{3}$$

Also,

$$\theta \leq \operatorname{tr} p_k(\boldsymbol{A}) \qquad \text{and} \qquad \Theta \leq \operatorname{tr} p_k(\boldsymbol{A}),$$

where $\Theta = \lim_{\ell \to \infty} \sqrt[\ell]{\alpha(G^{\boxtimes \ell})}$ is the Shannon capacity of a graph.

Proof. Let U be a k-independent set of G with $r = |U| = \alpha_k(G)$ vertices (first columns and rows of $\boldsymbol{A}$ corresponding to the vertices in U). Let $\boldsymbol{S}$ be the normalized characteristic matrix of this partition. Then, the quotient matrix of $p(\boldsymbol{A})$, $\boldsymbol{B}_k = \boldsymbol{S}^T p(\boldsymbol{A})\boldsymbol{S}$, is

$$\begin{pmatrix} \frac{1}{r}\sum_{u\in U}(p_k(\boldsymbol{A}))_{uu} & p_k(\theta_0) - \frac{1}{r}\sum_{u\in U}(p_k(\boldsymbol{A}))_{uu} \\ \frac{rp_k(\theta_0)-\sum_{u\in U}(p(\boldsymbol{A}))_{uu}}{n-r} & p_k(\theta_0) - \frac{rp_k(\theta_0)-\sum_{u\in U}(p(\boldsymbol{A}))_{uu}}{n-r} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{n}\sum_{i=0}^{d} m_i p_k(\theta_i) & 1 - \frac{1}{n}\sum_{i=0}^{d} m_i p_k(\theta_i) \\ \frac{r-\frac{r}{n}\sum_{i=0}^{d} m_i p_k(\theta_i)}{n-r} & 1 - \frac{r-\frac{r}{n}\sum_{i=0}^{d} m_i p_k(\theta_i)}{n-r} \end{pmatrix},$$

with eigenvalues $\mu_1 = p(\theta_0) = 1$ and

$$\mu_2 = \operatorname{tr}\boldsymbol{B}_k - 1 = w(p_k) - \frac{r - rw(p_k)}{n-r}.$$

where $w(p_k) = \frac{1}{n}\sum_{i=0}^{d} m_i p_k(\theta_i)$. Then, by interlacing, we have

$$0 \le \mu_2 \le w(p_k) - \frac{r - rw(p_k)}{n-r},$$

and the result follows. $\square$

The case $k = 1$.

As mentioned above, α_1 coincides with the standard independence number α. In this case the minor polynomial is $p_1(x) = \frac{x-\theta_d}{\theta_0-\theta_d}$. Then, (3) gives

$$\alpha_1 = \alpha \leq \operatorname{tr} p_1(\boldsymbol{A}) = \frac{-n\theta_d}{\theta_0 - \theta_d}, \tag{4}$$

which is Hoffman's bound in (1).

The case $k = 2$.

We already stated that $p_2(x) = \frac{(x-\theta_i)(x-\theta_{i-1})}{(\theta_0-\theta_i)(\theta_0-\theta_{i-1})}$. Then, (3) yields

$$\alpha_2 \le \operatorname{tr} p_2(\boldsymbol{A}) = n\frac{\theta_0 + \theta_i\theta_{i-1}}{(\theta_0 - \theta_i)(\theta_0 - \theta_{i-1})}, \tag{5}$$

in agreement with the result in (i).

Some examples

To compare the above bounds with those obtained before, let us consider again the Hamming graph $H(2,7)$ and the Johnson graph $J(14,7)$.

k	1	2	3	4	5	6	7
Bound from (2)	109	72	36	19	7	2	–
Bound from (**??**) ($k > 2$)	-	-	65	67	64	65	64
Bound from (i)-(iii)	-	21	56	6	55	3	55
Bound from (3)	**64**	**16**	**8**	**3**	**2**	**2**	**1**

Table: Comparison of the bounds for α_k in the Hamming graph $H(2,7)$.

k	3	4	5	6	7
Bound from (2)	464	125	20	2	–
Bound from (**??**)	935	721	546	408	302
Bound from (ii)-(iii)	26	10	5	3	2
Bound from (iv)	80	86	25	2	1
Bound from (3)	**19**	**6**	**2**	**2**	**1**

Table: Comparison of bounds for α_k in the Johnson graph $J(14, 7)$.

Note that the bounds for $k = 6, 7$ are equal to the correct values $\alpha_6 = 2$ (since both graphs are 2-antipodal, and $\alpha_7 = 1$ (since their diameter is $D = 7$). Besides, in the case of the Hamming graph, $\alpha_2 = 16$ since it contains the perfect Hamming code $H(7, 4)$.

An infinite family where our bound for α_{d-1} is tight

Assume that the minor polynomial takes non-zero value only at θ_1. Thus, $p_{d-1}(x) = \frac{1}{\prod_{i=2}^{d}(\theta_0-\theta_i)} \prod_{i=2}^{d}(x-\theta_i)$. Then, the bound of (3) is

$$\sum_{i=0}^{d} m_i p_{d-1}(\theta_i) = m_0 p_{d-1}(\theta_0) + m_1 p_{d-1}(\theta_1)$$

$$= 1 + \frac{\prod_{i=2}^{d}(\theta_1-\theta_i)}{\prod_{i=2}^{d}(\theta_0-\theta_i)} = 1 + m_1 \frac{\pi_1}{\pi_0}$$

where, in general, $\pi_i = \prod_{j=0, j\neq i} |\theta_i - \theta_j|$ for $i \in [0, d]$.
It is known (F., 1997) that G is an r-antipodal distance-regular graph if and only if its eigenvalue multiplicities are $m_i = \pi_0/\pi_i$ for i even, and $m_i = (r-1)\pi_0/\pi_i$ for i odd. Then, we get

$$\alpha_{d-1} \leq 1 + m_1 \frac{\pi_1}{\pi_0} = r,$$

which is the correct value.

4.3 The general case

The spectra of G^k and G are NOT related

Main result I: Inertial-type bound

$W(p) := \max_{u\in V}\{(p(\boldsymbol{A}))_{uu}\}$
$w(p) := \min_{u\in V}\{(p(\boldsymbol{A}))_{uu}\}$

(Abiad, Coutinho and F. 2019)
Let $p \in \mathbb{R}_k[x]$ with corresponding parameters $W(p)$, $w(p)$. Then,

$$\alpha_k \leq \min\{|\{i : p(\lambda_i) \geq w(p)\}|,\ |\{i : p(\lambda_i) \leq W(p)\}|.$$

(Abiad, Cioabă and Tait 2016)
Let G be a graph on n vertices. Then,

$$\alpha_k(G) \leq |\{i : \lambda_i^k \geq w_k(G)\}| \quad \text{and} \quad \alpha_k(G) \leq |\{i : \lambda_i^k \leq W_k(G)\}|.$$

Main result I: Inertial-type bound

$W(p) := \max_{u \in V}\{(p(\boldsymbol{A}))_{uu}\}$
$w(p) := \min_{u \in V}\{(p(\boldsymbol{A}))_{uu}\}$

(Abiad, Coutinho and F. 2019)
Let $p \in \mathbb{R}_k[x]$ with corresponding parameters $W(p)$, $w(p)$. Then,

$$\alpha_k \leq \min\{|\{i : p(\lambda_i) \geq w(p)\}|,\ |\{i : p(\lambda_i) \leq W(p)\}|.$$

Interesting if one can come up with a good choice for $p \in \mathbb{R}_k[x]$ or with an efficient method (like MILP) to compute it in practice

Main result II: Hoffman-type bound

$W(p) := \max_{u\in V}\{(p(\boldsymbol{A}))_{uu}\}$
$w(p) := \min_{u\in V}\{(p(\boldsymbol{A}))_{uu}\}$
$\Lambda(p) := \max_{i\in[2,n]}\{p(\lambda_i)\}$
$\lambda(p) := \min_{i\in[2,n]}\{p(\lambda_i)\}$

(Abiad, Coutinho and F. 2019)
Let G be a regular graph with n vertices and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $p \in \mathbb{R}_k[x]$ with corresponding parameters $W(p)$ and $\lambda(p)$, and assume $p(\lambda_1) > \lambda(p)$. Then,

$$\alpha_k \leq n\frac{W(p) - \lambda(p)}{p(\lambda_1) - \lambda(p)}.$$

Proof idea

Before we used a generalization of the Expander Mixing lemma, but we could have also used eigenvalue interlacing with quotient matrix.

Proof idea

- Let U be a k-independent set of G with $r = |U| = \alpha_k(G)$ vertices (first columns and rows of $\boldsymbol{A}$ corresponding to the vertices in U).
- Consider the partition wrt $\{U, V \setminus U\}$ with normalized characteristic matrix S.
- The quotient matrix of $p(\boldsymbol{A})$ is

$$\boldsymbol{S}^T p(\boldsymbol{A})\boldsymbol{S} = \boldsymbol{B}_k = \begin{pmatrix} \frac{1}{r}\sum\limits_{u\in U}(p(\boldsymbol{A}))_{uu} & p(\lambda_1) - \frac{1}{r}\sum_{u\in U}(p(\boldsymbol{A}))_{uu} \\ \frac{rp(\lambda_1) - \sum_{u\in U}(p(\boldsymbol{A}))_{uu}}{n-r} & p(\lambda_1) - \frac{rp(\lambda_1) - \sum_{u\in U}(p(\boldsymbol{A}))_{uu}}{n-r} \end{pmatrix}$$

  with eigenvalues $\mu_1 = p(\lambda_1)$ and

$$\mu_2 = \mathrm{tr}\boldsymbol{B}_k - p(\lambda_1) = \frac{1}{r}\sum_{u\in U}(p(\boldsymbol{A}))_{uu} - \frac{rp(\lambda_1) - \sum_{u\in U}(p(\boldsymbol{A}))_{uu}}{n-r}.$$

- By Haemers interlacing

$$\lambda(p) \le \mu_2 \le W(p) - \frac{rp(\lambda_1) - rW(p)}{n-r},$$

  solve for r and use $p(\lambda_1) - \lambda(p) > 0$. $\square$

Main result II: Hoffman-type bound

(Abiad, Coutinho and F. 2019)
Let G be a regular graph with n vertices and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $p \in \mathbb{R}_k[x]$ with corresponding parameters $W(p)$ and $\lambda(p)$, and assume $p(\lambda_1) > \lambda(p)$. Then,

$$\alpha_k \leq n \frac{W(p) - \lambda(p)}{p(\lambda_1) - \lambda(p)}.$$

Improvement of the Hoffman-type bound from
(Abiad, Cioabă and Tait 2016)

Sharp for some values of k
(see table coming in a few slides)

Corollary Hoffman-type bound, $k = 1$

Take p as any linear polynomial satisfying $p(\lambda_1) > \lambda(p)$, say $p(x) = x$.

Then, $W(p) = 0$, $p(\lambda_1) = \lambda_1$, $\lambda(p) = p(\lambda_n) = \lambda_n$.

(Hoffman 1970)
If G is regular then

$$\alpha_1 = \alpha \leq n \frac{-\lambda_n}{\lambda_1 - \lambda_n}.$$

Some other particular cases

Theorem (Abiad, Coutinho, F. (2018))

Let G be a δ-regular graph with n vertices and distinct eigenvalues $\theta_0(=\delta) > \theta_1 > \cdots > \theta_d$. Let $W_k = W(p) = \max_{u\in V}\{\sum_{i=1}^{k}(\boldsymbol{A}^k)_{uu}\}$.

- (i) *If $k=2$, then $\alpha_2 \le n\frac{\theta_0+\theta_i\theta_{i-1}}{(\theta_0-\theta_i)(\theta_0-\theta_{i-1})}$, where θ_i is the largest eigenvalue not greater than -1.*
- (ii) *If $k>2$ is odd, then $\alpha_k(G) \le n\frac{W_k - \sum_{j=0}^{k}\theta_d^j}{\sum_{j=0}^{k}\delta^j - \sum_{j=0}^{k}\theta_d^j}$.*
- (iii) *If $k>2$ is even, then $\alpha_k(G) \le n\frac{W_k+1/2}{\sum_{j=0}^{k}\delta^j+1/2}$.*
- (iv) *If $G=(V,E)$ is a walk-regular graph, then $\alpha_k(G) \le n\frac{1-\lambda(q_k)}{q_k(\delta)-\lambda(q_k)}$ for $k=0,\dots,d-1$, where $q_k = p_0+\cdots+p_k$ with the p_i's being the predistance polynomials of G, and $\lambda(q_k) = \min_{i\in[2,d]}\{q_k(\theta_i)\}$.*

Bounds computational comparison: k=2

Graph	our bound 2016	Lovász ϑ_2	our Hoffman-like 2019	α_2
Balaban 10-cage	32	**17**	**17**	17
Frucht graph	6	**3**	**3**	3
Meredith graph	20	10	14	10
Moebius-Kantor graph	8	**4**	**4**	4
Gosset graph	2	2	2	2
Balaban 11-cage	41	26	27	time
Gray graph	33	**11**	**11**	11
Nauru graph	10	**6**	**6**	6
Blanusa first snark graph	8	**4**	**4**	4
Pappus graph	9	**3**	**3**	3
Perkel graph	12	**5**	**5**	5
Bucky ball	23	12	14	12
Heawood graph	2	2	2	2
Hoffman graph	6	**2**	**2**	2
Coxeter graph	13	**7**	**7**	**7**
Desargues graph	10	**5**	**5**	4
Tutte-Coxeter graph	10	**6**	**6**	6
Tutte graph	21	10	11	10
Flower snark	7	**5**	**5**	5
Wells graph	6	**3**	**3**	2
Foster graph	44	**22**	**22**	21
Hexahedron	2	2	2	2
Dodecahedron	9	**4**	**4**	4
Icosahedron	2	2	2	2
Balaban 10-cage	32	**17**	**17**	17
Frucht graph	6	**3**	**3**	3
Meredith Graph	20	10	14	10
Moebius-Kantor Graph	8	**4**	**4**	4
Dodecahedron	9	**4**	**4**	4
...	...	...	...	...

Next natural question

$p \in \mathbb{R}_k[x]$, U a k-independent set in G

$\Downarrow$

$p(A)$ has a principal submatrix defined by U that is diagonal, with diagonal entries defined by a linear combination of various closed walks

$\Downarrow$

In our bounds $p(A) = A^k$

$\Downarrow$

Other polynomials? How to choose $p(A)$ to optimize our bounds?

Further optimization?

Using MILP techniques to implement the eigenvalue bounds, and sometimes assuming extra properties on the graph, yes!

Joint work with A. Abiad, G. Coutinho, B.D. Nogueira and S. Zeijlemaker

Main questions

Inertial-type bound: best polynomial for general k?

Hoffman-type bound: best polynomial for general k?

Optimizing the inertial-type bound

(Abiad, Coutinho and F. 2019)

Let G be a graph with n vertices and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $p \in \mathbb{R}_k[x]$ with corresponding parameters $W(p)$ and $\lambda(p)$.

$$\alpha_k \leq \min\{|i : p(\lambda_i) \geq w(p)|, |i : p(\lambda_i) \leq W(p)|\}.$$

Optimizing the inertial-type bound

Let G have spectrum $\{\theta_0^{m_0}, \dots, \theta_d^{m_d}\}$. Let $p(x) = a_k x^k + \cdots + a_0$, $\mathbf{b} = (b_0, \dots, b_d) \in \{0,1\}^{d+1}$, and $\mathbf{m} = (m_0, \dots, m_d)$.

Variables: $a_0, \dots, a_k$ and $b_1, \dots, b_d$.

For each $u \in V(G)$, we run one MILP and find the best objective value of all:

$$
\boxed{
\begin{array}{rl}
\texttt{minimize} & \boldsymbol{m}^\top \boldsymbol{b} \\
\texttt{subject to} & \sum_{i=0}^k a_i (A^i)_{vv} \ge 0, \quad v \in V(G) \setminus \{u\} \\
 & \sum_{i=0}^k a_i (A^i)_{uu} = 0 \\
 & \sum_{i=0}^k a_i {\theta_j}^i - M b_j + \epsilon \le 0, \quad j = 0, \dots, d \quad (*) \\
 & \boldsymbol{b} \in \{0,1\}^{d+1}
\end{array}
}
$$

Each $b_j = 1$ represents an index j so that $p(\theta_j) \ge w(p) = 0$. Condition $(*)$ gives that $p(\theta_j) \ge 0$ implies $b_j = 1$.
So, upon minimizing the quantity of such indices j, we are optimizing $p(x)$ and the corresponding bound $\alpha_k \le \mathbf{m}^\top \mathbf{b}$. For each $u \in V(G)$, we write one such MILP and find the best objective value of all.

Optimizing the inertial-type bound: extra assumption

Can we avoid running the MILP for each $u \in V(G)$?

Yes, assuming k-partially walk-regularity.

G is k-**partially walk-regular**, for some integer $k \geq 0$, if the number of closed walks of a given length $l \leq k$, rooted at a vertex v, only depends on l.
Every graph is k-partially walk-regular for $k = 0, 1$, and every regular graph is 2-partially walk-regular.

Optimizing the inertial-type bound: extra assumption

$$
\begin{array}{ll}
\texttt{minimize} & \boldsymbol{m}^\top \boldsymbol{b} \\
\texttt{subject to} & \sum_{i=0}^{k} a_i (A^i)_{vv} \geq 0, \quad v \in V(G) \setminus \{u\} \\
& \sum_{i=0}^{k} a_i (A^i)_{uu} = 0 \\
& \sum_{i=0}^{k} a_i {\theta_j}^i - M b_j + \epsilon \leq 0, \quad j = 0, ..., d \quad (*) \\
& \boldsymbol{b} \in \{0,1\}^{d+1}
\end{array}
$$

In the case of k-**partially walk-regular** graphs, we only need to run the MILP once, since all vertices have the same number of closed walks of length smaller of equal than k.

For instance, for Odd graphs the MILP finds the best polynomials for upper bounding α_k.

An alternative inertial-type bound

(Abiad, Coutinho, F., Nogueira, and Zeijlemaker 2020)
Let G be a k-partially walk-regular graph with adjacency matrix eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $p_k \in \mathbb{R}_k[x]$ such that $\sum_{i=1}^{n} p_k(\lambda_i) = 0$. Then,

$$\chi_k \geq 1 + \max\left(\frac{|j : p_k(\lambda_j) < 0|}{|j : p_k(\lambda_j) > 0|}\right).$$

Optimizing an alternative inertial-type bound

(Abiad, Coutinho, F., Nogueira and Zeijlemaker 2020)
Let G be a k-partially walk-regular graph with adjacency matrix eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $p_k \in \mathbb{R}_k[x]$ such that $\sum_{i=1}^n p_k(\lambda_i) = 0$. Then,

$$\chi_k \geq 1 + \max\left(\frac{|j : p_k(\lambda_j) < 0|}{|j : p_k(\lambda_j) > 0|}\right).$$

$$\begin{array}{rll}
\texttt{maximize} & 1 + \frac{n - \mathbf{1}^\top \boldsymbol{b}}{\ell} & \\
\texttt{subject to} & \boxed{\sum_{j=1}^n \sum_{i=0}^k a_i \lambda_j^i = 0} & \\
& \sum_{i=0}^k a_i \lambda_j^i - M b_j + \epsilon \leq 0, & j = 1, \ldots, n \\
& \sum_{i=0}^k a_i \lambda_j^i - M c_j \leq 0, & j = 1, \ldots, n \\
& \sum_{i=1}^n c_i = \ell & \\
& \boldsymbol{b} \in \{0,1\}^n, \quad \boldsymbol{c} \in \{0,1\}^n &
\end{array}$$

Trace p(A)=0

Optimizing an alternative inertial-type bound

(Abiad, Coutinho, F., Nogueira and Zeijlemaker 2020)
Let G be a k-partially walk-regular graph with adjacency matrix eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $p_k \in \mathbb{R}_k[x]$ such that $\sum_{i=1}^{n} p_k(\lambda_i) = 0$. Then,

$$\chi_k \geq 1 + \max\left(\frac{|j : p_k(\lambda_j) < 0|}{|j : p_k(\lambda_j) > 0|}\right).$$

$$\begin{array}{lll} \texttt{maximize} & 1 + \frac{n - \mathbf{1}^\top \boldsymbol{b}}{\ell} & \\ \texttt{subject to} & \sum_{j=1}^{n} \sum_{i=0}^{k} a_i \lambda_j^i = 0 & \\ & \sum_{i=0}^{k} a_i \lambda_j^i - M b_j + \epsilon \leq 0, & j = 1, \ldots, n \\ & \boxed{\sum_{i=0}^{k} a_i \lambda_j^i - M c_j \leq 0}, & j = 1, \ldots, n \\ & \sum_{i=1}^{n} c_i = \ell & \\ & \boldsymbol{b} \in \{0,1\}^n, \quad \boldsymbol{c} \in \{0,1\}^n & \end{array}$$

$$p(\lambda_j) > 0 \implies c_j = 1$$

Open problems

- Same MILP methods could be useful to find the target polynomial in other graphs and/or for other values of k.
- Relationship between inertial-type and ratio-type bounds via the obtained polynomials from the MILPs?
- Application of the bounds for α_k to the non existence of perfect codes?
- (Lenstra 1983) showed that MILP with fixed number of variables are polynomial solvable. Given a fixed n, find an (efficient) algorithm to compute the best bounds with the MILPs?

Gràcies per l'atenció
Thanks for your attention