New Lower Bounds on Multicolour Diagonal Ramsey Numbers

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Reading Group: New trends in Combinatorics

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Based on work by Conlon & Ferber; Widgerson.

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Diagonal Ramsey Numbers:

$$R(t; \ell) := R(\underbrace{t, t, \ldots, t}_{\ell \text{ copies}})$$

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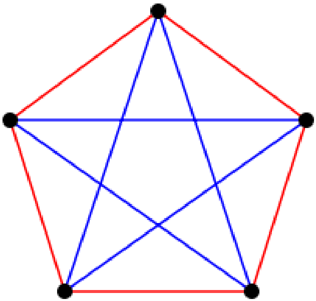

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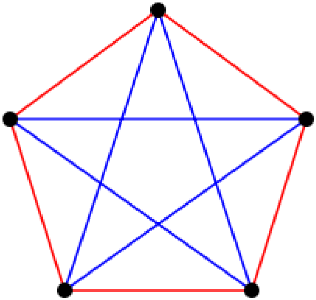

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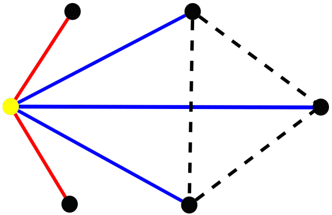

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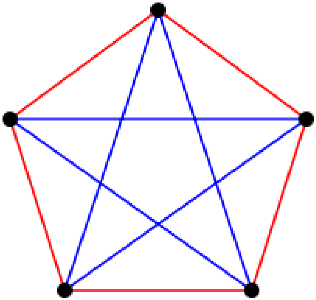

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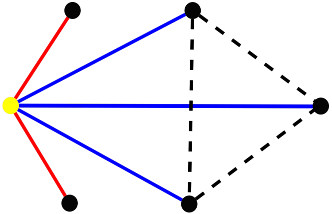

One of the dashed lines is blue or all are red.

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This allows us to deduce that there is at least one graph on $\sqrt{2}^t$ vertices with no monochromatic K_t.

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At least t of the v_i share a colour and therefore form a monochromatic K_t.

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Gives lower bound essentially $R(t;\ell) \geq 3^{\frac{\ell t}{6}}$.

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Corollary

$R(t;\ell) \geq 2^{7\ell t/24+o(t)} (\gg 3^{\ell t/6})$.

Host Graph: Let q be prime, and suppose that $t \neq 0 \mod q$.
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That is, we take a random induced subgraph of V of size n and shall show it contains no monochromatic clique of size t.

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This has eigenvalues $i(s-1)$ (multiplicity 1) and $-i$ (multiplicity $s-1$). So if $s \neq 1 \mod q$, M is non-singular over $\mathbb{F}_q$ and thus $\alpha = 0$ so $v_1, \ldots, v_s$ is a linearly independent set of vectors whereby $s \leq t = \dim(\mathbb{F}_q^t)$.

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If $s = 1 \mod q$, the same argument with $v_1, \ldots, v_{s-1}$ yields $s-1 \leq t$ and $s-1 \neq t$ as then $t = 0 \mod q$ which we assumed was not the case.

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Expression above increasing in r for $r \leq t/2$ so max attained when $r = t/2$.

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Potential cliques of rank r.

Expression above increasing in r for $r \le t/2$ so max attained when $r = t/2$.

So sum over all ranks gives that we have at most $N_t = q^{\frac{5t^2}{8} + o(t^2)}$ potential cliques.

Conlon-Ferber Proof 3/3

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This completes the proof of the theorem.

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Arguing similarly to before we can deduce that provided N is sufficiently small, there are no monochromatic copies of K_t.